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Difference L operators related to q-characters

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Abstract
We introduce a factorized difference operator $L(u)$ annihilated by the Frenkel–Reshetikhin screening operator for the quantum affine algebra $U_q\big(C_n^{(1)}\big)$. We identify the coefficients of $L(u)$ with the fundamental q-characters, and establish a number of formulae for their higher analogues. They include Jacobi–Trudi and Weyl-type formulae, cancelling tableau sums, Casorati determinant solution to the T-system, and so forth. Analogous operators for the orthogonal series $U_q\big(B_n^{(1)}\big)$ and $U_q\big(D_n^{(1)}\big)$ are also presented.

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1. Introduction

In this paper we introduce a factorized difference operator $L(u)$ related to the quantum affine algebra $U_q\big(C_n^{(1)}\big)$, and present its application to the q-characters of some finite-dimensional representations. As for basic facts on finite-dimensional representations, we refer to [CP1, CP2, Ka].

The theory of q-characters was introduced in [FR2, FM] motivated by their study of deformed $\mathcal{W}$-algebras (see also [Kn]). The q-characters χ_q are Laurent polynomials in infinitely many variables $\big\{Y_a(u)^{\pm 1}\big| 1 \leqslant a \leqslant n, u \in \mathbb{C}\big\}$, which reduce to linear combinations of the usual characters with respect to a classical simple subalgebra in the limit $q \to 1$. For an irreducible representation V, $\chi_q(V)$ contains the highest weight monomial with coefficient 1, and the rest are generated by multiplying lowering factors corresponding to the negative roots. One of the fundamental properties of the q-characters is that they enjoy a symmetry analogous to simple reflections in the Weyl group [FR2, FM]. It is represented as $S_a\chi_q = 0$ for all $1 \leqslant a \leqslant n$, where S_a is called the screening operator.

We construct a difference operator $L(u)$ acting on functions of a variable u with the following features:

- $S_a L(u) = 0$ for all a.
- $L(u)$ generates all the fundamental q-characters of $U_q\big(C_n^{(1)}\big)$.

By fundamental q-characters we mean those for the fundamental representations in the sense of [CP1]. Such an operator of order n was known for $A_n^{(1)}$ in [FR1, FRS]. Curiously, our $L(u)$ for $C_n^{(1)}$ is of order $2n+2$ and has a form similar to the $A_{2n+1}^{(1)}$ case, which contrasts with the well-known embedding $C_n \hookrightarrow A_{2n-1}$. It contains each fundamental q-character twice and admits a factorization into $2n$ products of first-order difference operators and a second-order one. The last piece is further factorized if one extends the coefficient field $\mathcal{Y} = \mathbb{Z}\big[Y_a(u)^{\pm 1}\big]_{1\leqslant a\leqslant n, u\in\mathbb{C}}$ to $\mathbb{Z}\big[Q_a(u)^{\pm 1}\big]_{1\leqslant a\leqslant n, u\in\mathbb{C}}$ by introducing the Baxter Q functions as (2.1). Such an identification is based on the connection [FR2] between q-characters and the analytic Bethe ansatz [R, KS, KOS] for solvable lattice models [B].

To utilize the ideas in the latter is another motivation of the paper. Roughly, χ_q corresponds to an eigenvalue formula for transfer matrices, and the condition $\chi_q \in \cap_a \mathrm{Ker} S_a$ to its pole-freeness. Our approach here is based on the difference equation $L(u)w(u) = 0$, and involves only elementary linear algebra and a portion of combinatorics. By means of a difference analogue of the Wronskian method for ordinary linear differential equations, we express the fundamental q-characters and their higher analogues in terms of a ratio of Casorati determinants. They may be viewed as analogues of the Weyl formula for the usual characters. By a standard argument [NNSY], the ratio of the Casorati determinants is equal to a sum over semistandard tableaux on letters $\{1, 2, \ldots, 2n+2\}$. Upto this point, parallel results are also derivable for the $A_{2n+1}^{(1)}$ case. A curiosity for the $C_n^{(1)}$ case is that one of the tableau variables has a minus sign (see (2.5)). Nevertheless all the negative contributions cancel out for many examples including fundamental q-characters as exhibited in appendix A. It will be interesting to seek applications of the present results in light of the previous studies [BHK, CK, DDT, FRS, KLWZ, M2, SS, S1].

This paper is organized as follows. In section 2, we define $L(u)$ and derive various formulae for the fundamental q-characters. In section 3 we extend the results in the preceding section slightly to a class generated by $L(u)$. This is partly motivated by a similar structure observed for the Stokes multipliers [DDT, S1, S2]. In section 4, we present a solution of the $C_n^{(1)}$ T-system [KNS] in terms of the Casorati determinants. An analogous result for $A_n^{(1)}$ is available in [KLWZ]. In section 5 we give similar difference operators $L(u)$ for $B_n^{(1)}$ and $D_n^{(1)}$, which originate in [KOS, TK]. However, their orders are not finite as opposed to the $C_n^{(1)}$ case. Appendix A contains a combinatorial proof of proposition 2.4. Appendix B provides some basic lemmas connecting the Casorati determinants, tableau sums and Jacobi–Trudi-type formulae.

2. $L(u)$ and fundamental q-characters

Let $\{\alpha_a \mid 1 \leqslant a \leqslant n\}$ and $\{\Lambda_a \mid 1 \leqslant a \leqslant n\}$ be the sets of simple roots and fundamental weights of C_n normalized as $(\alpha_a \mid \alpha_a) = 1 + \delta_{an}$. Throughout the paper we set

$$N = 2n+2.$$

We denote the Frenkel–Reshetikhin variable Y_{a,q^u} [FR2] for $U_q\big(C_n^{(1)}\big)$ by $Y_a(u) (1 \leqslant a \leqslant n)$, and introduce the Baxter Q functions $Q_a(u)$ related to $Y_a(u)$ as

$$Y_a(u) = \frac{Q_a\left(u - \frac{(\alpha_a|\alpha_a)}{2}\right)}{Q_a\left(u + \frac{(\alpha_a|\alpha_a)}{2}\right)}. \tag{2.1}$$

For the definition of the q-character we refer to the original paper [FR2]. Here we recall the screening operator S_a, which sends the elements in $\mathcal{Y} = \mathbb{Z}\big[Y_a(u)^{\pm 1}\big]_{1\leqslant a\leqslant n, u\in\mathbb{C}}$ to the ring extended by adjoining the extra symbols $\{S_a(u) \mid 1 \leqslant a \leqslant n, u \in \mathbb{C}\}$. The action is given by

$$S_a Y_b(u) = \delta_{ab} Y_b(u) S_b(u)$$

and the Leibniz rule. Thus, for example $S_a Y_b(u)^{-1} = -\delta_{ab} Y_b(u)^{-1} S_b(u)$. The symbol $S_a(u)$ is assumed to obey the following relation in the extended ring:

$$\begin{aligned} S_a(u + (\alpha_a \mid \alpha_a)) &= A_a\left(u + \frac{(\alpha_a \mid \alpha_a)}{2}\right) S_a(u) \\ A_a(u) &= \prod_{b=1}^{n} \frac{Q_b(u - (\alpha_a \mid \alpha_b))}{Q_b(u + (\alpha_a \mid \alpha_b))} \end{aligned} \tag{2.2}$$

where $A_a(u)$ can actually be expressed in terms of the Y-variables only.

Let

$$J = \{1 \prec 2 \prec \cdots \prec n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}\}$$

be an ordered set. In what follows we shall work with the two sets of variables $\{z_a(u) \mid a \in J\}$ and $\{x_a(u) \mid 1 \leqslant a \leqslant N\}$ specified by

$$z_a(u) = \frac{Y_a\left(u + \frac{a}{2}\right)}{Y_{a-1}\left(u + \frac{a+1}{2}\right)} \qquad z_{\bar{a}}(u) = \frac{Y_{a-1}\left(u + \frac{2n-a+3}{2}\right)}{Y_a\left(u + \frac{2n-a+4}{2}\right)} \qquad 1 \leqslant a \leqslant n \tag{2.3}$$

$$x_a(u) = z_a(u) \qquad x_{2n+3-a}(u) = z_{\bar{a}}(u) \qquad 1 \leqslant a \leqslant n \tag{2.4}$$

$$x_{n+1}(u) = -x_{n+2}(u) = \frac{Q_n\left(u + \frac{n}{2}\right) Q_n\left(u + \frac{n+4}{2}\right)}{Q_n\left(u + \frac{n+2}{2}\right)^2} \tag{2.5}$$

where $Y_0(u) = 1$. Note that unlike (2.3) and (2.4), $x_{n+1}(u) = -x_{n+2}(u)$ is *not* expressible as a ratio of Y_n.

In this paper, we denote the q-character $\chi_q\left(V_{\Lambda_a}(q^u)\right)$ of the fundamental representation $V_{\Lambda_a}(q^u)$ by $T_1^{(a)}(u)$ $(1 \leqslant a \leqslant n)$. In terms of the variables (2.3), it is given by

$$T_1^{(a)}\left(u + \frac{1}{2}\right) = \sum z_{i_1}\left(u + \frac{a-1}{2}\right) z_{i_2}\left(u + \frac{a-3}{2}\right) \cdots z_{i_a}\left(u - \frac{a-1}{2}\right) \tag{2.6}$$

where the sum runs over $i_1, \ldots, i_a \in J$ such that

$$i_1 \prec \cdots \prec i_a \tag{2.7}$$

$$\text{if } i_k = c, i_l = \bar{c} \text{ for some } 1 \leqslant c \leqslant n, \text{ then } n + k - l \geqslant c. \tag{2.8}$$

Upon a suitable convention adjustment, this agrees with $s_a\left(zq^{-1/2}\right)$ in section 11.3 of [FR1]. It also coincides with $\Lambda_1^{(a)}(u + 1/2)$ in [KS], where the condition (2.7), (2.8) was first introduced under which the number of summands is indeed equal to the dimension $\binom{2n}{a} - \binom{2n}{a-2}$ of the ath fundamental representation of C_n. We have $T_1^{(a)}(u) = Y_a(u) + \cdots$, where $Y_a(u)$ is the highest weight monomial [FM].

Example 2.1. For $n = 2$, the two fundamental q-characters read

$$T_1^{(1)}(u) = Y_1(u) + \frac{Y_2\left(u + \frac{1}{2}\right)}{Y_1(u+1)} + \frac{Y_1(u+2)}{Y_2\left(u + \frac{5}{2}\right)} + \frac{1}{Y_1(u+3)}$$

$$T_1^{(2)}(u) = Y_2(u) + \frac{Y_1\left(u + \frac{1}{2}\right) Y_1\left(u + \frac{3}{2}\right)}{Y_2(u+2)} + \frac{Y_1\left(u + \frac{1}{2}\right)}{Y_1\left(u + \frac{5}{2}\right)} + \frac{Y_2(u+1)}{Y_1\left(u + \frac{3}{2}\right) Y_1\left(u + \frac{5}{2}\right)} + \frac{1}{Y_2(u+3)}.$$

Let D be a difference operator $Dg(u) = g(u+1)D$. We use the notation

$$\overrightarrow{\prod_{i=1}^{k}} X_i = X_1 X_2 \cdots X_k \qquad \overleftarrow{\prod_{i=1}^{k}} X_i = X_k \cdots X_2 X_1.$$

By a direct calculation one finds

Lemma 2.2.

$$\overrightarrow{\prod_{a=1}^{n}} (1 - z_{\bar{a}}(u)D) \left(1 - z_{\bar{n}}(u) z_n(u+1) D^2\right) \overleftarrow{\prod_{a=1}^{n}} (1 - z_a(u)D)$$

$$= \overrightarrow{\prod_{a=1}^{n}} (z_a(u+n+1-a) - D) \left(z_{\bar{n}}(u-1) z_n(u) - D^2\right)$$

$$\times \overleftarrow{\prod_{a=1}^{n}} (z_{\bar{a}}(u-n-2+a) - D)$$

$$= \overleftarrow{\prod_{i=1}^{N}} (1 - \epsilon_i x_i(u) D) = - \overrightarrow{\prod_{i=1}^{N}} (\epsilon_i x_i(u+n+1-i) - D)$$

where $\epsilon_i = 1$ *for* $i \neq n+1, n+2$ *and* $\epsilon_{n+1} = \epsilon_{n+2} = \pm 1$.

In particular the middle quadratic factor can be factorized as

$D^2 - z_{\bar{n}}(u-1) z_n(u) = D^2 - x_{n+1}(u-1) x_{n+1}(u) = (x_{n+1}(u) \pm D)(x_{n+2}(u-1) \pm D)$.

We define the L operator by

$$L(u) = \overrightarrow{\prod_{i=1}^{N}} (x_i(u+n+1-i) - D) = - \overleftarrow{\prod_{i=1}^{N}} (1 - x_i(u) D). \qquad (2.9)$$

Due to lemma 2.2, $L(u)$ is a polynomial in D of order N with coefficients in $\mathfrak{Y}$. For a difference operator of the form $\sum_j c_j(u) D^j$ with $c_j(u) \in \mathfrak{Y}$, the screening operator acts as

$$S_a\left(\sum_j c_j(u) D^j\right) = \sum_j (S_a c_j(u)) D^j.$$

Proposition 2.3.

$$L(u) Q_1(u) = 0 \qquad (2.10)$$

$$S_a L(u) = 0 \qquad 1 \leqslant a \leqslant n. \qquad (2.11)$$

Proof. The rightmost factor in the first expression of (2.9) reads $(Q_1(u+1)/Q_1(u) - D)$, proving (2.10). As for (2.11), we illustrate the $a = n$ case. By the definition, S_n acts non-trivially only on the middle four factors in (2.9), which is expanded as $\left(v = u + \frac{n}{2}\right)$

$$\frac{Y_{n-1}\left(v - \frac{1}{2}\right)}{Y_{n-1}\left(v + \frac{3}{2}\right)} - \left(\frac{Y_{n-1}\left(v + \frac{1}{2}\right)}{Y_n(v+2)} + \frac{Y_n(v)}{Y_{n-1}\left(v + \frac{3}{2}\right)}\right) D$$

$$+ \left(\frac{Y_{n-1}\left(v + \frac{5}{2}\right)}{Y_n(v+3)} + \frac{Y_n(v+1)}{Y_{n-1}\left(v + \frac{3}{2}\right)}\right) D^3 + D^4.$$

Upon applying S_a, this becomes

$$\left(S_n(v+2)\frac{Y_{n-1}\left(v+\frac{1}{2}\right)}{Y_n(v+2)} - S_n(v)\frac{Y_n(v)}{Y_{n-1}\left(v+\frac{3}{2}\right)}\right)D + \left(-S_n(v+3)\frac{Y_{n-1}\left(v+\frac{5}{2}\right)}{Y_n(v+3)} + S_n(v+1)\frac{Y_n(v+1)}{Y_{n-1}\left(v+\frac{3}{2}\right)}\right)D^3.$$

This vanishes under (2.2), i.e.,

$$S_n(v+2) = \frac{Y_n(v)Y_n(v+2)}{Y_{n-1}\left(v+\frac{3}{2}\right)Y_{n-1}\left(v+\frac{1}{2}\right)}S_n(v). \qquad \square$$

Let us introduce the notation:

$$\begin{aligned}
X_u(i_1,\ldots,i_a) &= \prod_{k=1}^{a} x_{i_k}(u+1-k)\\
Z_u(i_1,\ldots,i_a) &= \prod_{k=1}^{a} z_{i_k}(u+1-k)\\
\sum_i X_u(i_1,\ldots,i_a) &= \sum_{1\leqslant i_1<\cdots<i_a\leqslant N} X_u(i_1,\ldots,i_a)\\
\sum_i Z_u(i_1,\ldots,i_a) &= \sum_{1\preceq i_1\prec\cdots\prec i_a\preceq\bar{1}} Z_u(i_1,\ldots,i_a).
\end{aligned} \tag{2.12}$$

Expanding (2.9) we get the two expressions:

$$\begin{aligned}
L(u) &= \sum_{a=0}^{N}(-1)^a\left(\sum_i X_{u+n}(i_1,\ldots,i_{N-a})\right)D^a\\
&= -\sum_{a=0}^{N}(-1)^a\left(\sum_i X_{u+a-1}(i_1,\ldots,i_a)\right)D^a.
\end{aligned}$$

Thus we find

$$L(u) = \sum_{i=n+2}^{N}(-1)^i e_{N-i}\left(u+\frac{i}{2}\right)D^i - \sum_{i=0}^{n}(-1)^i e_i\left(u+\frac{i}{2}\right)D^i \tag{2.13}$$

$$e_a\left(u+\frac{a}{2}\right) = -e_{N-a}\left(u+\frac{a}{2}\right) \qquad 0\leqslant a\leqslant N \tag{2.14}$$

$$e_a\left(u+\frac{a}{2}\right) := \sum_i X_{u+a-1}(i_1,\ldots,i_a). \tag{2.15}$$

In particular (2.14) tells that $e_{n+1}(u) = 0, e_N(u) = -1$. In view of (2.5), the sum (2.15) contains sign factors. However, they all cancel out leaving the 'positive' contributions only.

Proposition 2.4. (Cancelling tableau sums for fundamental q-characters). *We have $T_1^{(a)}(u) = e_a(u)$, namely,*

$$T_1^{(a)}(u) = \sum_i X_{u+\frac{a}{2}-1}(i_1,\ldots,i_a) \qquad 1\leqslant a\leqslant n. \tag{2.16}$$

The proof is available in appendix A, where we show that the cancellation in (2.15) precisely leaves the sum in (2.6)–(2.8). From proposition 2.4 and (2.13), one has

Theorem 2.5.

$$L(u) = \sum_{i=n+2}^{N} (-1)^i T_1^{(N-i)}\left(u + \frac{i}{2}\right) D^i - \sum_{i=0}^{n} (-1)^i T_1^{(i)}\left(u + \frac{i}{2}\right) D^i.$$

From proposition 2.3 and theorem 2.5 we conclude $S_a T_1^{(b)}(u) = 0$ for all $1 \leqslant a, b \leqslant n$. For later convenience, we extend $T_1^{(a)}(u)$ to $a \in Z$ by

$$\begin{aligned} &T_1^{(a)}(u) + T_1^{(N-a)}(u) = 0 \qquad \forall a \\ &T_1^{(a)}(u) = 0 \quad \text{for } a < 0 \quad T_1^{(0)}(u) = 1. \end{aligned} \tag{2.17}$$

Then theorem 2.5 is rephrased as

$$L(u) = -\sum_{i=0}^{N} (-1)^i T_1^{(i)}\left(u + \frac{i}{2}\right) D^i. \tag{2.18}$$

Now we turn to the linear difference equation

$$L(u) w(u) = 0. \tag{2.19}$$

Let $w_1(u), \ldots, w_N(u)$ be a basis of the solutions of (2.19). For $i_1, \ldots, i_m \in \mathbb{Z} (m \leqslant N)$, we prepare a shorthand for the Casorati determinant:

$$[i_1, \ldots, i_m] = \det \begin{pmatrix} w_1(u+i_1) & \cdots & w_1(u+i_m) \\ \vdots & & \vdots \\ w_m(u+i_1) & \cdots & w_m(u+i_m) \end{pmatrix}. \tag{2.20}$$

We will write $[0, \ldots, 3]$ to mean the consecutive filling $[0, 1, 2, 3]$ for example. Note that u-dependence is suppressed in LHS and the overall shift $i_r \to i_r + 1$ is equivalent to $u \to u+1$. From (2.19) it follows that

$$[0, \ldots, N-1] = -[1, \ldots, N]. \tag{2.21}$$

Proposition 2.6. (Weyl-type formula for fundamental q-characters).

$$T_1^{(a)}\left(u + \frac{a}{2}\right) = \frac{[0, \ldots, a-1, a+1, \ldots, N]}{[1, 2, \ldots, N]} \qquad 0 \leqslant a \leqslant N.$$

Proof. Solve the simultaneous equation obtained by setting $w = w_1, \ldots, w_N$ in (2.19), i.e.,

$$w(u+N) = \sum_{i=0}^{N-1} (-1)^i T_1^{(i)}\left(u + \frac{i}{2}\right) w(u+i) \tag{2.22}$$

with respect to the coefficients $T_1^{(i)}\left(u + \frac{i}{2}\right)$. □

Proposition 2.6 is also shown by applying proposition B.3 to proposition 2.4. For the choice $w(u) = Q_1(u)$ (see (2.10)), (2.22) is often called the 'T–Q relation'.

Before concluding the section, we make a few miscellaneous remarks on $L(u)^{-1}$. For $m \in \mathbb{Z}_{\geqslant 1}$ define

$$T_m^{(1)}\left(u + \frac{1}{2}\right) = \sum z_{i_1}\left(u - \frac{m-1}{2}\right) z_{i_2}\left(u - \frac{m-3}{2}\right) \cdots z_{i_m}\left(u + \frac{m-1}{2}\right)$$

where the sum runs over $i_1, \ldots, i_m \in J$ such that

$$i_k \preceq i_{k+1} \qquad \text{or} \qquad (i_k, i_{k+1}) = (\bar{n}, n) \qquad \text{for} \quad 1 \leqslant k \leqslant m-1. \tag{2.23}$$

We put $T_1^{(0)}(u) = T_0^{(a)}(u) = 1$. From the first expression in lemma 2.2, we find

$$-L(u)^{-1} = \sum_{m=0}^{\infty} T_m^{(1)}\Big(u + \frac{m}{2}\Big) D^m \tag{2.24}$$

hence $S_a T_m^{(1)}(u) = 0$.

The horizontal tableaux obeying condition (2.23) have appeared in equation (3.15a) of [KS]. We suppose that the quantity $T_m^{(1)}(u)$ is the irreducible q-character with highest weight monomial $\prod_{j=1}^m Y_1\big(u + \frac{m+1-2j}{2}\big)$. Multiplying (2.18) and (2.24) we deduce two types of 'T–T relations':

$$\sum_{a=0}^{N} (-1)^a T_{m-a}^{(1)}\Big(u - \frac{a}{2}\Big) T_1^{(a)}\Big(u + \frac{m-a}{2}\Big) = \delta_{m0}$$
$$\sum_{a=0}^{N} (-1)^a T_{m-a}^{(1)}\Big(u + \frac{m+a}{2}\Big) T_1^{(a)}\Big(u + \frac{a}{2}\Big) = \delta_{m0}$$

for $m \in \mathbb{Z}_{\geqslant 0}$. As is well known for the A_n case, the T–Q relation (2.10), namely,

$$\sum_{a=0}^{N} (-1)^a Q_1(u+a) T_1^{(a)}\Big(u + \frac{a}{2}\Big) = 0$$

is obtained from the limit $m \to \infty$ in the latter T–T relations with a formal identification

$$Q_1(u) = \lim_{m\to\infty} T_m^{(1)}\Big(u + \frac{m}{2}\Big).$$

The T–T relations are also obtainable by expanding the determinant expression for $T_m^{(1)}(u)$ in remark 4.4 with respect to the first row or mth column.

3. Higher q-characters generated by $L(u)$

Evaluation of the ratio $\frac{[0,i_1,i_2,\ldots,i_{N-1}]}{[0,1,2,\ldots,N-1]}$ of the Casorati determinants has been done in appendix B. Especially, proposition B.3 is an essential result saying that it is a polynomial in the fundamental q-characters $\{T_1^{(a)}(u) \,|\, 1 \leqslant a \leqslant n, u \in \mathbb{C}\}$, hence annihilated by all the screening operators S_a. Clarifying its q-character content (irreducibility, decomposition into classical characters, etc) for general $i_1, \ldots, i_{N-1}$ is left to a future study. See remark 3.4. In this section we concentrate on a modest class generated by repeated application of (2.22). The resulting relation of the form

$$w(u+k) = \sum_{i=0}^{N-1} (-1)^i H_k^{(i)}\Big(u + \frac{i}{2}\Big) w(u+i) \qquad k \geqslant 0 \tag{3.1}$$

uniquely determines the coefficients. In other words, we define $H_k^{(i)}(u)$ $(0 \leqslant i \leqslant N-1, k \in \mathbb{Z}_{\geqslant 0}, u \in \mathbb{C})$ via the recursion relation and the initial condition:

$$\begin{aligned} &H_{k+1}^{(i)}\Big(u + \frac{i}{2}\Big) = -T_1^{(i)}\Big(u + \frac{i}{2}\Big) H_k^{(N-1)}\Big(u + \frac{N+1}{2}\Big) - H_k^{(i-1)}\Big(u + \frac{i+1}{2}\Big) \\ &H_0^{(i)}(u) = \delta_{i0} \end{aligned} \tag{3.2}$$

where in the former we understand that $H_k^{(-1)}(u) = 0$. By the definition $H_k^{(i)}(u) = (-1)^i \delta_{ik}$ for $0 \leqslant k \leqslant N-1$ and $H_N^{(i)}(u) = T_1^{(i)}(u)$. The following shows that the class $H_k^{(i)}(u)$ is relevant to Young diagrams of hook shape.

Proposition 3.1. (Jacobi–Trudi and Weyl-type formulae).

$$H_k^{(i)}\left(u+\frac{i}{2}\right) = \begin{cases} \delta_{ik}(-1)^i & 0 \leqslant k \leqslant N-1 \\ -\det_{1\leqslant j,l\leqslant k-N+1}\left(T_1^{(\lambda'_j-j+l)}\left(u+\dfrac{N-2-\lambda'_j+j+l}{2}\right)\right) & k \geqslant N \end{cases}$$
$$= -\frac{[0,1,\ldots,i-1,i+1,i+2,\ldots,N-1,k]}{[0,\ldots,N-1]}$$

where λ'_j is specified from i and N by $\lambda'_j = 1+(N-i-1)\delta_{j1}$.

Proof. Due to (2.17), the determinant satisfies the same recursion as (3.2), proving the first expression. As for the second one, solve the simultaneous equation obtained by taking $w = w_1, \ldots, w_N$ in (3.1). It can also be derived by rewriting the first one by using proposition B.3. □

Regarding $\lambda' = (\lambda'_j)$ as the transpose of the Young diagram λ (cf [M1]), we see that the latter is of hook shape with width $k-N+1$ and depth $N-i$.

Let us rewrite (2.22) and (3.1) into matrix form. We introduce the N-dimensional vectors and square matrices

$$\vec{w}(u) = \begin{pmatrix} w(u) \\ w(u+1) \\ \vdots \\ w(u+N-1) \end{pmatrix}$$
$$\vec{h}_k(u) = \begin{pmatrix} H_k^{(0)}(u) \\ -H_k^{(1)}\left(u+\frac{1}{2}\right) \\ \vdots \\ (-1)^{N-1}H_k^{(N-1)}\left(u+\frac{N-1}{2}\right) \end{pmatrix}$$
$$\mathcal{T}(u) = \begin{pmatrix} 0 & 0 & \cdots & 0 & T_1^{(0)}(u) \\ 1 & 0 & & 0 & -T_1^{(1)}\left(u+\frac{1}{2}\right) \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & 1 & 0 & (-1)^{N-2}T_1^{(N-2)}\left(u+\frac{N-2}{2}\right) \\ 0 & \cdots & 0 & 1 & (-1)^{N-1}T_1^{(N-1)}\left(u+\frac{N-1}{2}\right) \end{pmatrix}$$
$$\mathcal{H}_k(u) = (\vec{h}_k(u), \vec{h}_{k+1}(u), \ldots, \vec{h}_{k+N-1}(u)).$$

Then (2.22) and (3.1) lead to

$${}^t\vec{w}(u+1) = {}^t\vec{w}(u)\mathcal{T}(u) \qquad {}^t\vec{w}(u+k) = {}^t\vec{w}(u)\mathcal{H}_k(u).$$

Therefore we have a product formulae

$$\mathcal{T}(u)\mathcal{T}(u+1)\cdots\mathcal{T}(u+k-1) = \mathcal{H}_k(u).$$

See [DDT, S1, S2] for a similar structure observed for the Stokes multipliers in the $A_n^{(1)}$ case. We set

$$\sigma_j = \begin{cases} 1 & 1 \leqslant j \leqslant n \\ -1 & n+1 \leqslant j \leqslant N. \end{cases} \tag{3.3}$$

Conjecture 3.2. *For $k \geqslant N+1$, the quantity $\sigma_i H_k^{(i)}\big(u+\frac{i}{2}\big)$ $(0 \leqslant i \leqslant N-1)$ is the irreducible q-character with the highest weight monomial $(Y_0(u) = 1)$*

$$\begin{array}{ll} Y_n\left(u+\dfrac{n+2}{2}\right)\displaystyle\prod_{j=2}^{k-N} Y_1\left(u+\dfrac{N+2j-1}{2}\right) & \text{if} \quad i = n+1 \\ Y_{\min(i,N-i)}\left(u+\dfrac{i}{2}\right)\displaystyle\prod_{j=1}^{k-N} Y_1\left(u+\dfrac{N+2j-1}{2}\right) & \textit{otherwise}. \end{array}$$

Let us turn to the C_n content of $H_k^{(i)}$. Let $(\alpha \mid \gamma)$ be the Young diagram of hook shape with width $\alpha+1$ and depth $\gamma+1$ (Frobenius notation [M1]). The corresponding C_n character with highest weight $\alpha\Lambda_1 + \Lambda_{\gamma+1}$ is represented by $\chi_{(\alpha|\gamma)}$. Especially, we shall denote the character of the trivial representation by $\chi_{(-1|0)} = 1$ rather than by $\chi_{(0|-1)}$. Recall the homomorphism [FR2]

$$\beta\colon \mathcal{Y} = \mathbb{Z}\big[Y_a(u)^{\pm 1}\big]_{1\leqslant a\leqslant n, u\in\mathbb{C}} \to \mathbb{Z}[\mathrm{e}^{\pm\Lambda_a}]_{1\leqslant a\leqslant n}$$

sending $Y_a(u)^{\pm 1}$ to $\mathrm{e}^{\pm\Lambda_a}$. For $1 \leqslant i \leqslant N-1$ we know

$$\beta\Big(\sigma_i T_1^{(i)}(u)\Big) = \begin{cases} 0 & \text{if} \quad i = n+1 \\ \chi_{(0|\min(i,N-i)-1)} & \text{otherwise.} \end{cases}$$

Proposition 3.3. *For $k \geqslant N+1$, the image of $H_k^{(i)}(u)$ $(0 \leqslant i \leqslant N-1)$ under β is given as*

$$\beta\Big(H_k^{(0)}\Big) = -\beta\Big(H_{k-1}^{(N-1)}\Big) = \sum^{\wedge} \chi_{(k-N-2j-1|0)}$$

$$\beta\Big(H_k^{(a)}\Big) = \begin{cases} \displaystyle\sum^{\vee} \chi_{(k-N-2j|a-1)} + \sum^{\vee} \chi_{(k-N-2j-1|a)} & 1 \leqslant a \leqslant n-1 \\ \displaystyle -\sum^{\vee} \chi_{(k-N-2j|N-a-1)} - \sum^{\vee} \chi_{(k-N-2j-1|N-a-2)} & n+2 \leqslant a \leqslant N-2 \end{cases}$$

$$\beta\Big(H_{k-1}^{(n)}\Big) = -\beta\Big(H_k^{(n+1)}\Big) = \sum^{\vee} \chi_{(k-N-2j-1|n-1)}$$

where we have suppressed u on the LHS since the result is independent of it. The sum $\sum^{\wedge} \chi(\alpha-2j \mid \gamma)$ $\big($respectively $\sum^{\vee} \chi(\alpha-2j \mid \gamma)\big)$ extends over $j \in \mathbb{Z}_{\geqslant 0}$ such that $\alpha - 2j \geqslant \min(0, \gamma-1)$ (respectively $\alpha - 2j \geqslant 0$).

Proof. Check relation (3.2) under β by means of $(1 \leqslant a \leqslant n,\ p \geqslant 0)$

$$\chi_{(p-1|0)}\chi_{(0|a-1)} = \chi_{(p|a-1)} + \chi_{(p-1|a)} + \chi_{(p-1|a-2)} + \chi_{(p-2|a-1)}$$

where $\chi_{(p|a)} = 0$ for any p if $a \leqslant -1$ or $a = n$. □

Extending β on $\mathcal{Y}$, we introduce a map β' which is applicable also to the solutions $w_i(u)$ of (2.19) as follows. We parametrize the fundamental weight Λ_a in terms of the orthogonal basis ε_b $(1 \leqslant b \leqslant n)$ as $\Lambda_a = \varepsilon_1 + \cdots + \varepsilon_a$. The C_n Weyl group acts as a permutation or $\times(\pm 1)$ on $\{\varepsilon_i\}$ as is well known. For $1 \leqslant i \leqslant N$, introduce the variable x_i by

$$x_i = \begin{cases} 1 & i = n+1 \\ -1 & i = n+2 \\ \mathrm{e}^{\sigma_i \varepsilon_{\min(i,N+1-i)}} & \text{otherwise.} \end{cases}$$

We take β' to be the same as β on $\mathcal{Y}$, and

$$\beta'\colon \quad x_i(u) \mapsto x_i \qquad w_i(u) \mapsto x_{N+1-i}^{u} \qquad 1 \leqslant i \leqslant N.$$

The former ($i \neq n+1, n+2$) follows from (2.3) and (2.4) by applying β. The latter has been adjusted to (B.3). As a result we get

$$\beta' : \quad \frac{[0, i_1, i_2, \ldots, i_{N-1}]}{[0, 1, 2, \ldots, N-1]} \mapsto \frac{\det_{1\leqslant j,k\leqslant N}\left(x_j^{i_{k-1}}\right)}{\det_{1\leqslant j,k\leqslant N}\left(x_j^{k-1}\right)}$$

where $i_0 = 0$. This is a Weyl group invariant Laurent polynomial in $\mathrm{e}^{\Lambda_1}, \ldots, \mathrm{e}^{\Lambda_n}$, hence a linear combination of C_n characters. Under β', proposition 3.1 yields Jacobi–Trudi and Weyl-type formulae (but A_{2n+1}-like rather than C_n) for the linear combinations of C_n characters associated with the hook diagrams.

Remark 3.4. For any $i_1, \ldots, i_{N-1} \in \mathbb{Z}$, one has $\frac{[0,i_1,i_2,\ldots,i_{N-1}]}{[0,1,2,\ldots,N-1]} \in \mathfrak{Y}$. However, the coefficients of the monomials in $\mathfrak{Y}$ are not always all positive or negative. For example, for C_2, one has

$$\begin{aligned}\frac{[0,1,3,4,6,7]}{[0,1,2,3,4,5]} &= T_1^{(1)}\left(u+\frac{3}{2}\right)T_1^{(1)}\left(u+\frac{5}{2}\right) - T_1^{(2)}(u+1)T_1^{(2)}(u+3)\\ &= \frac{Y_1\left(u+\frac{7}{2}\right)}{Y_1\left(u+\frac{11}{2}\right)} - Y_2(u+2)Y_2(u+4) + Y_2(u+3) - \cdots\end{aligned}$$

which consists of 19 monomials.

4. Solution of the *T*-system

The T-system is a set of functional relations among a certain family $\left\{T_m^{(a)}(u)\middle| 1 \leqslant a \leqslant n, m \in \mathbb{Z}_{\geqslant 1}, u \in \mathbb{C}\right\}$ of commuting transfer matrices in solvable lattice models proposed in [KNS] for any $U_q\left(X_n^{(1)}\right)$. It has the form of the Toda field equation on a discrete spacetime:

$$T_m^{(a)}\left(u - \frac{(\alpha_a \mid \alpha_a)}{2}\right)T_m^{(a)}\left(u + \frac{(\alpha_a \mid \alpha_a)}{2}\right) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + S_m^{(a)}(u)$$

where $m \geqslant 1$, $T_0^{(a)}(u) = 1$ and $S_m^{(a)}(u)$ is a certain product of $T_m^{(a)}(u)$. α_a ($1 \leqslant a \leqslant n$) denotes the simple root of X_n. The T-system uniquely determines $T_m^{(a)}(u)$ as a rational function of the fundamental ones $\left\{T_1^{(a)}(u)\middle|\ 1 \leqslant a \leqslant n, u \in \mathbb{C}\right\}$. Moreover it has been proved for non-exceptional algebras [KNH] that $T_m^{(a)}(u)$ is actually a polynomial in these variables expressed as a determinant or Pfaffian.

Let $W_m^{(a)}(u)$ denote the Kirillov–Reshetikhin module over $U_q\left(X_n^{(1)}\right)$ [KR]. By this we mean the irreducible one whose q-character has the highest weight monomial $\prod_{j=1}^m Y_a\left(u + \frac{(\alpha_a|\alpha_a)}{2}(m+1-2j)\right)$. In light of the correspondence between the transfer matrices and q-characters (cf section 6.1 in [FR2]), it is natural to make

Conjecture 4.1 *The identification* $T_m^{(a)}(u) = \chi_q\left(W_m^{(a)}(u)\right)$ *solves the* T*-system.*

Motivated by these aspects, here we present the solution of the $C_n^{(1)}$ T-system in terms of the Casorati determinants (2.20), which may be viewed as a Weyl-type formulae for q-characters of the Kirillov–Reshetikhin module. The T-system is explicitly given by

$$T_m^{(a)}\left(u-\tfrac{1}{2}\right)T_m^{(a)}\left(u+\tfrac{1}{2}\right) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \qquad 1 \leqslant a \leqslant n-2 \tag{4.1}$$

$$T_{2m}^{(n-1)}\left(u-\tfrac{1}{2}\right)T_{2m}^{(n-1)}\left(u+\tfrac{1}{2}\right) = T_{2m+1}^{(n-1)}(u)T_{2m-1}^{(n-1)}(u) + T_{2m}^{(n-2)}(u)T_m^{(n)}\left(u-\tfrac{1}{2}\right)T_m^{(n)}\left(u+\tfrac{1}{2}\right) \tag{4.2}$$

$$T^{(n-1)}_{2m+1}\big(u-\tfrac{1}{2}\big)T^{(n-1)}_{2m+1}\big(u+\tfrac{1}{2}\big)=T^{(n-1)}_{2m+2}(u)T^{(n-1)}_{2m}(u)+T^{(n-2)}_{2m+1}(u)T^{(n)}_{m}(u)T^{(n)}_{m+1}(u) \tag{4.3}$$

$$T^{(n)}_{m}(u-1)T^{(n)}_{m}(u+1)=T^{(n)}_{m+1}(u)T^{(n)}_{m-1}(u)+T^{(n-1)}_{2m}(u). \tag{4.4}$$

where $T^{(a)}_0(u)=T^{(0)}_m(u)=1$.

Set

$$\xi^{(a)}_m(u)=[0,1,\ldots,a-1,a+m,a+m+1,\ldots,N+m-1]$$
$$\xi(u)=\xi^{(1)}_0(u)=[0,1,\ldots,N-1].$$

Lemma 4.2. *$\xi^{(a)}_m(u)$ satisfies the relations:*

$$\xi^{(a)}_m(u)\xi^{(a)}_m(u+1)-\xi^{(a)}_{m+1}(u)\xi^{(a)}_{m-1}(u+1)-\xi^{(a+1)}_m(u)\xi^{(a-1)}_m(u+1)=0 \tag{4.5}$$

$$\xi^{(a)}_m(u)=(-1)^{a-\frac{N}{2}+m}\xi^{(N-a)}_m\left(u+a-\frac{N}{2}\right). \tag{4.6}$$

Proof. Equation (4.5) is a Plücker relation. Equation (4.6) is shown by applying (2.17) to the latter formula in proposition B.3. □

From the $a=N/2$ case of (4.6), it follows that

$$\xi^{(n+1)}_m(u)=0 \qquad m\in 2\mathbb{Z}_{\geqslant 0}+1.$$

Consequently, one has the identities:

$$\begin{aligned}\xi^{(n)}_{2m}(u)\xi^{(n+2)}_{2m}(u-1)&=\xi^{(n+1)}_{2m}(u)\xi^{(n+1)}_{2m}(u-1)\\ \xi^{(n)}_{2m+1}(u)\xi^{(n+2)}_{2m+1}(u-1)&=-\xi^{(n+1)}_{2m}(u)\xi^{(n+1)}_{2m+2}(u-1)\end{aligned} \tag{4.7}$$

since the difference of LHS and RHS become $-\xi^{(n+1)}_{2m+1}(u-1)\xi^{(n+1)}_{2m-1}(u)$ and $\xi^{(n+1)}_{2m+1}(u)\xi^{(n+1)}_{2m+1}(u-1)$, respectively, due to (4.5).

Now our solution is given by

Proposition 4.3. *The following solves the $C^{(1)}_n$ T-system.*

$$T^{(a)}_m\left(u+\frac{a+m-1}{2}\right)=(-1)^m\frac{\xi^{(a)}_m(u)}{\xi(u)} \qquad 1\leqslant a\leqslant n-1 \tag{4.8}$$

$$T^{(n)}_m\left(u+\frac{n+2m}{2}\right)T^{(n)}_m\left(u+\frac{n+2m-2}{2}\right)=\frac{\xi^{(n)}_{2m}(u)}{\xi(u)} \tag{4.9}$$

$$T^{(n)}_m\left(u+\frac{n+2m}{2}\right)T^{(n)}_{m+1}\left(u+\frac{n+2m}{2}\right)=\frac{\xi^{(n)}_{2m+1}(u)}{\xi(u+1)} \tag{4.10}$$

$$T^{(n)}_m\left(u+\frac{n+2m}{2}\right)^2=\frac{\xi^{(n+1)}_{2m}(u)}{\xi(u)} \tag{4.11}$$

which are equivalent, due to (4.6), to the alternative forms:

$$T^{(a)}_m\left(u+\frac{a+m-1}{2}\right)=(-1)^{a-\frac{N}{2}}\frac{\xi^{(N-a)}_m\big(u+a-\frac{N}{2}\big)}{\xi(u)} \qquad 1\leqslant a\leqslant n-1 \tag{4.12}$$

$$T^{(n)}_m\left(u+\frac{n+2m}{2}\right)T^{(n)}_m\left(u+\frac{n+2m-2}{2}\right)=\frac{\xi^{(n+2)}_{2m}(u-1)}{\xi(u+1)} \tag{4.13}$$

$$T^{(n)}_m\left(u+\frac{n+2m}{2}\right)T^{(n)}_{m+1}\left(u+\frac{n+2m}{2}\right)=\frac{\xi^{(n+2)}_{2m+1}(u-1)}{\xi(u+1)}. \tag{4.14}$$

Proof. First we are to show the consistency of (4.9), (4.13) and (4.11). Namely, evaluation of $\left(T_m^{(n)}(v)T_m^{(n)}(v-1)\right)^2$ by (4.9) and (4.13) indeed coincides with the result by (4.11). To see this, we multiply (4.9) not with itself but with (4.13), leading to $\left(v = u + \frac{n+2m}{2}\right)$

$$\left(T_m^{(n)}(v)T_m^{(n)}(v-1)\right)^2 = \frac{\xi_{2m}^{(n)}(u)\xi_{2m}^{(n+2)}(u-1)}{\xi(u)\xi(u-1)}.$$

On the other hand, (4.11) says that the LHS is equal to

$$\frac{\xi_{2m}^{(n+1)}(u)\xi_{2m}^{(n+1)}(u-1)}{\xi(u)\xi(u-1)}.$$

Thus they agree owing to the first relation in (4.7). Similarly, the consistency of (4.10), (4.14) and (4.11) is confirmed by means of the second relation in (4.7).

Now we proceed to check (4.1)–(4.4). Upon substituting (4.8), the difference between the LHS and RHS of (4.1) vanishes due to (4.5). Similarly, one can verify (4.2) and (4.3) by using (4.9) and (4.10). As for the last relation (4.4), we first multiply $\left(T_m^{(n)}(u)\right)^2$ and regroup the factors as

$$\begin{aligned}&\left(T_m^{(n)}(u)T_m^{(n)}(u-1)\right)\left(T_m^{(n)}(u+1)T_m^{(n)}(u)\right)\\ &\qquad = \left(T_m^{(n)}(u)T_{m+1}^{(n)}(u)\right)\left(T_{m-1}^{(n)}(u)T_m^{(n)}(u)\right) + \left(T_m^{(n)}(u)\right)^2 T_{2m}^{(n-1)}(u).\end{aligned}$$

Upon applying (4.9), (4.10) and (4.11) to the first, second and third terms, respectively, the result again reduces to the Plücker relation (4.5). □

Combining (4.9), (4.10), (4.13) and (4.14), we also have the expression

$$\begin{aligned}T_m^{(n)}\left(u + \frac{n+2m}{2}\right) &= (-1)^m \prod_{j=1}^{m} \frac{\xi_{2j-1}^{(n)}(u+1)}{\xi_{2j-2}^{(n)}(u+1)}\\ &= \prod_{j=1}^{m} \frac{\xi_{2j-1}^{(n+2)}(u)}{\xi_{2j-2}^{(n+2)}(u)}.\end{aligned}$$

This implies the square root of (4.11) is taken so that $T_m^{(n)}(u) = \prod_{j=1}^m Y_n(u+m+1-2j) + \cdots$.

Remark 4.4. The following Jacobi–Trudi-type formula is also known as theorem 3.1 in [KNH]:

$$\begin{aligned}T_m^{(a)}(u) &= \det_{1\leqslant j,l\leqslant m}\left(T_1^{(a-j+l)}\left(u + \frac{j+l-m-1}{2}\right)\right) \qquad 1 \leqslant a \leqslant n-1\\ T_m^{(n)}(u) &= (-1)^m \operatorname*{pf}_{1\leqslant j,l\leqslant 2m}\left(T_1^{(n+1-j+l)}\left(u + \frac{j+l-2m-1}{2}\right)\right)\end{aligned}$$

where pf stands for the Pfaffian.

5. $B_n^{(1)}$ and $D_n^{(1)}$ cases

Here we present L operators having the same property as proposition 2.3 for $B_n^{(1)}$ and $D_n^{(1)}$. However they are not polynomials in D. Essentially they are the generating series of the pole-free combinations in the analytic Bethe ansatz [KOS, TK].

For $B_n^{(1)}$ we set

$$z_a(u) = \frac{Y_a(u+a)}{Y_{a-1}(u+a+1)} \qquad z_{\bar{a}}(u) = \frac{Y_{a-1}(u+2n-a)}{Y_a(u+2n-a+1)} \qquad 1 \leqslant a \leqslant n-1$$

$$z_n(u) = \frac{Y_n\left(u+\frac{2n+1}{2}\right) Y_n\left(u+\frac{2n-1}{2}\right)}{Y_{n-1}(u+n+1)} \qquad z_{\bar{n}}(u) = \frac{Y_{n-1}(u+n)}{Y_n\left(u+\frac{2n+3}{2}\right) Y_n\left(u+\frac{2n+1}{2}\right)}$$

$$z_0(u) = \frac{Y_n\left(u+\frac{2n-1}{2}\right)}{Y_n\left(u+\frac{2n+3}{2}\right)}$$

$$L(u) = \overrightarrow{\prod_{a=1}^{n}} \left(1 - z_{\bar{a}}(u)D^2\right)\left(1+z_0(u)D^2\right)^{-1} \overleftarrow{\prod_{a=1}^{n}} \left(1 - z_a(u)D^2\right).$$

Let α_a $(1 \leqslant a \leqslant n)$ be the simple root of B_n normalized as $(\alpha_a \mid \alpha_a) = 2 - \delta_{an}$. We let S_a denote the screening operator for $B_n^{(1)}$ specified by (2.2) and (2.1).

Proposition 5.1. $S_a L(u) = 0$ *for all* $1 \leqslant a \leqslant n$.

Proof. For $a \neq n$ this can be directly checked as (2.11). For $a = n$, we are to show $S_n X(u+n-2) = 0$, where

$$X(v) = \left(1 - \frac{Y_{n-1}(v+2)}{Y_n(v+5/2)Y_n(v+7/2)}D^2\right)\left(1 + \frac{Y_n(v+3/2)}{Y_n(v+7/2)}D^2\right)^{-1} \times \left(1 - \frac{Y_n(v+3/2)Y_n(v+5/2)}{Y_{n-1}(v+3)}D^2\right).$$

Expanding $X(v)$ one has

$$X(v) = 1 - f(v)D^2 + h(v)\sum_{j=0}^{\infty}(-1)^j k(v+2j)D^{2j+4}$$

$$f(v) = \frac{Y_n(v+3/2)}{Y_n(v+7/2)} + \frac{Y_{n-1}(v+2)}{Y_n(v+5/2)Y_n(v+7/2)} + \frac{Y_n(v+3/2)Y_n(v+5/2)}{Y_{n-1}(v+3)}$$

$$k(v) = \frac{1}{Y_n(v+11/2)} + \frac{Y_n(v+9/2)}{Y_{n-1}(v+5)}$$

$$h(v) = Y_n(v+3/2) + \frac{Y_{n-1}(v+2)}{Y_n(v+5/2)}.$$

It is easy to verify $S_n f(v) = S_n k(v) = S_n h(v) = 0$. □

We introduce the expansion coefficients of $L(u)$ as

$$L(u) = 1 + \sum_{a \geqslant 1}(-1)^a T^a(u+a)D^{2a} \tag{5.1}$$

$$L(u)^{-1} = 1 + \sum_{m \geqslant 1} T_m(u+m)D^{2m}. \tag{5.2}$$

Under the correspondence (2.1), they agree with those defined in equation (2.7) in [KOS]. For $1 \leqslant a \leqslant n-1$, $T^a(u)$ essentially coincides with $s_a(z)$ in section 11.2 of [FR1], which may be viewed as the q-character of the ath fundamental representation of $U_q\left(B_n^{(1)}\right)$. Note also that $L(u)Q_1(u) = 0$. As a corollary of proposition 5.1, these coefficients are annihilated by all the screening operators S_a.

For $D_n^{(1)}$ we set

$$
\begin{aligned}
&z_a(u) = \frac{Y_a(u+a)}{Y_{a-1}(u+a+1)} \qquad z_{\bar{a}}(u) = \frac{Y_{a-1}(u+2n-a-1)}{Y_a(u+2n-a)} \qquad 1 \leqslant a \leqslant n-2 \\
&z_{n-1}(u) = \frac{Y_n(u+n-1)Y_{n-1}(u+n-1)}{Y_{n-2}(u+n)} \qquad z_{\overline{n-1}}(u) = \frac{Y_{n-2}(u+n)}{Y_n(u+n+1)Y_{n-1}(u+n+1)} \\
&z_n(u) = \frac{Y_n(u+n-1)}{Y_{n-1}(u+n+1)} \qquad z_{\bar{n}}(u) = \frac{Y_{n-1}(u+n-1)}{Y_n(u+n+1)}
\end{aligned}
\tag{5.3}
$$

$$
L(u) = \prod_{a=1}^{\overrightarrow{n}} \left(1 - z_{\bar{a}}(u)D^2\right)\left(1 - z_n(u)z_{\bar{n}}(u+2)D^4\right)^{-1} \prod_{a=1}^{\overleftarrow{n}} \left(1 - z_a(u)D^2\right).
$$

Let $\alpha_a (1 \leqslant a \leqslant n)$ be the simple root of D_n normalized as $(\alpha_a \mid \alpha_a) = 2$. We let S_a denote the screening operator for $D_n^{(1)}$ specified by (2.2) and (2.1).

Proposition 5.2. $S_a L(u) = 0$ *for all* $1 \leqslant a \leqslant n$.

The proof is similar to proposition 5.1. In particular, $S_n L(u) = 0$ is reduced to the following two lemmas.

Lemma 5.3. *Setting*

$$
h_a(u) := Y_a(u) + \frac{Y_{n-2}(u+1)}{Y_a(u+2)} \qquad k_a(u) := Y_a(u)^{-1} + \frac{Y_a(u-2)}{Y_{n-2}(u-1)} \qquad (a = n-1, n)
$$

one has $S_n h_n(u) = S_n k_n(u) = 0$.

Lemma 5.4. *The* Y_n*-dependent factors in (5.3) can be expanded as*

$$
\begin{aligned}
&\left(1 - \frac{Y_{n-2}(v+4)}{Y_{n-1}(v+5)Y_n(v+5)}D^2\right)\left(1 - \frac{Y_{n-1}(v+3)}{Y_n(v+5)}D^2\right)\left(1 - \frac{Y_n(v+3)}{Y_n(v+7)}D^4\right)^{-1} \\
&\times \left(1 - \frac{Y_n(v+3)}{Y_{n-1}(v+5)}D^2\right)\left(1 - \frac{Y_{n-1}(v+3)Y_n(v+3)}{Y_{n-2}(v+4)}D^2\right) \\
&\qquad = 1 - \sum_{j\geqslant 0}(k_{n-1}(v+4j+5)h_n(v+3) \\
&\qquad\quad + (1-\delta_{j,0})k_n(v+4j+5)h_{n-1}(v+3))D^{4j+2} \\
&\qquad\quad + \sum_{j\geqslant 0}\Bigg(k_{n-1}(v+4j+7)h_{n-1}(v+3) + k_n(v+4j+7)h_n(v+3) \\
&\qquad\quad - \delta_{j,0}\frac{Y_{n-2}(v+4)}{Y_{n-2}(v+6)}\Bigg)D^{4j+4}
\end{aligned}
$$

where $v = u + n - 4$.

Defining $T^a(u)$ by (5.1) and (5.3), one finds, under the correspondence (2.1), that $T^a(u)$ coincides with $\mathcal{T}^a(u)$ in equation (2.9) in [TK]. For $1 \leqslant a \leqslant n-2$, $T^a(u)$ essentially agrees with $s_a(z)$ in section 11.4 of [FR1], which may be viewed as the q-character of the ath fundamental representation of $U_q\left(D_n^{(1)}\right)$. Note also that $L(u)Q_1(u) = 0$. As a corollary of proposition 5.2, $T^b(u)$ is annihilated by all the screening operators S_a.

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Appendix A. Proof of proposition 2.4

We begin by grouping the summands on the RHS of (2.16) as

$$\sum_{i \not\ni n+1,n+2} + \sum_{i \ni n+1, i \not\ni n+2} + \sum_{i \ni n+2, i \not\ni n+1} + \sum_{i \ni n+1,n+2} .$$

Due to (2.5), the second and third terms cancel each other. The summands in the first and the last terms can be expressed with $z_i(u)$ by using (2.5) and

$$x_{n+1}(u)x_{n+2}(u-1) = -z_n(u)z_{\bar{n}}(u-1).$$

The result reads

$$\begin{aligned}\sum_i X_{u+\frac{a}{2}-1}(i_1,\ldots,i_a) \\ = \sum_i Z_{u+\frac{a}{2}-1}(i_1,\ldots,i_a) - \sum_{k=0}^{a-2}\sum_{i,j} Z_{u+\frac{a}{2}-1}(i_1,\ldots,i_k,n,\bar{n},j_1,\ldots,j_{a-2-k}).\end{aligned} \tag{A.1}$$

Here the first sum is taken according to (2.12). $\sum_{i,j}$ in the second term extends over $i_1,\ldots,i_k,j_1,\ldots,j_{a-2-k} \in J$ such that $1 \preceq i_1 \prec \cdots \prec i_k \preceq n$ and $\bar{n} \preceq j_1 \prec \cdots \prec j_{a-2-k} \preceq \bar{1}$. Note that the second term contains the summands with at most *two* n and *two* $\bar{n}$ in the i, j-arrays. So those patterns do not match a semistandard column tableau with respect to the order $\prec$. This will be the point that the remainder of the analysis concerns. From (A.1) and (2.7), (2.8), proposition 2.4 is reduced to $\left(v = u + \frac{a}{2} - 1\right)$

$$\sum_{k=0}^{a-2}\sum_{i,j} Z_v(i_1,\ldots,i_k,n,\bar{n},j_1,\ldots,j_{a-2-k}) = \sum_{1\preceq i_1\prec\cdots\prec i_a\preceq\bar{1},\ (2.8)\ \text{is broken}} Z_v(i_1,\ldots,i_a). \tag{A.2}$$

For $n = 2$ or $0 \leqslant a \leqslant 2$, it is straightforward to check (A.2). Thus we assume $n \geqslant 3$ and fix $3 \leqslant a \leqslant n$ (and any $v \in \mathbb{C}$) from now on. We call the array $(i_1,\ldots,i_a)$ of elements $i_1,\ldots,i_a \in J$ a tableau. We do not *a priori* assume (2.7) and (2.8).

Lemma A.1.

$$Z_v\left(\ldots,b,\overbrace{\cdots}^{n-b+1},\bar{b},\ldots\right) = Z_v\left(\ldots,b-1,\overbrace{\cdots}^{n-b+1},\overline{b-1},\ldots\right) \qquad 2 \leqslant b \leqslant n$$

with no change for ... *parts.*

Proof. For any u we have

$$z_b(u)z_{\bar{b}}(u-n+b-2) = z_{b-1}(u)z_{\overline{b-1}}(u-n+b-2) \qquad 2 \leqslant b \leqslant n. \tag{A.3}$$

□

Actually (A.3) is also valid for $b = 1$ if we interpret $z_0(u) = z_{\bar{0}}(u) = 1$.

We introduce a map of tableaux:

$$\tau_b: (i_1, \ldots, i_a) \mapsto (i'_1, \ldots, i'_a) \qquad 2 \leqslant b \leqslant n$$

where the RHS is obtained from the LHS by making the transformation

$$\Bigl(\ldots, b, \overbrace{\cdots}^{n-b+1}, \bar{b}, \ldots\Bigr) \mapsto \Bigl(\ldots, b-1, \overbrace{\cdots}^{n-b+1}, \overline{b-1}, \ldots\Bigr)$$

for all the $(b, \bar{b})$ pairs matching this configuration. When there is no such $(b, \bar{b})$ pair, we assume the action of τ_b is trivial, i.e., $(i', \ldots, i'_a) = (i_1, \ldots, i_a)$. Due to lemma A.1, τ_b is Z_v-preserving.

Example A.2. We set $n = a = 9$. Omitting the parenthesis we have

$$\begin{aligned}
3\,5\,7\,9\,9\,\bar{9}\,\bar{8}\,\bar{7}\,\bar{3} &\stackrel{\tau_9}{\longmapsto} 3\,5\,7\,8\,9\,\bar{8}\,\bar{8}\,\bar{7}\,\bar{3}\\
&\stackrel{\tau_8}{\longmapsto} 3\,5\,7\,7\,9\,\bar{8}\,\bar{7}\,\bar{7}\,\bar{3}\\
&\stackrel{\tau_7}{\longmapsto} 3\,5\,6\,6\,9\,\bar{8}\,\bar{6}\,\bar{6}\,\bar{3}\\
&\stackrel{\tau_6}{\longmapsto} 3\,5\,5\,6\,9\,\bar{8}\,\bar{6}\,\bar{5}\,\bar{3}\\
&\stackrel{\tau_5}{\longmapsto} 3\,4\,5\,6\,9\,\bar{8}\,\bar{6}\,\bar{4}\,\bar{3} &\qquad (A.4)\\
&\stackrel{\tau_3}{\longmapsto} 2\,4\,5\,6\,9\,\bar{8}\,\bar{6}\,\bar{4}\,\bar{2}. &\qquad (A.5)
\end{aligned}$$

On all the tableaux τ_2 and τ_4 act trivially.

Set

$$V = \{(i_1 \prec \cdots \prec i_k \preceq n, \bar{n} \preceq j_1 \prec \cdots \prec j_{a-2-k}) \mid 0 \leqslant k \leqslant a-2\} \tag{A.6}$$

$$W = \{(i_1 \prec \cdots \prec i_a) \mid (2.8) \text{ is broken}\} \tag{A.7}$$

where $i_r, j_r \in J$ extend over all the possibilities so that V (respectively W) coincides with the range of the sum on the LHS (respectively RHS) of (A.2). Since τ_b are Z_v-preserving, proposition 2.4 is reduced to constructing a bijection $\tau: V \longrightarrow W$ from their composition. This will be achieved in proposition A.8 later.

For $2 \leqslant b \leqslant n$ and $l, m \in \mathbb{Z}_{\geqslant 0}$ such that $|l - m| = 0, \pm 1$, $(l, m) \neq (0, 0)$, we introduce the subset $V_b^{l,m} \subset V$ defined by

$$V_b^{l,m} = \Bigl\{\Bigl(i_1 \prec \cdots \prec i_\alpha \prec \overbrace{b \cdots b}^{l} \prec j_1 \prec \cdots \prec j_\beta \prec \overbrace{\bar{b} \cdots \bar{b}}^{m}$$

$$\prec k_1 \prec \cdots \prec k_\gamma\Bigr) \Bigm| (\mathrm{A}), (\mathrm{B}), (\mathrm{C})\Bigr\} \tag{A.8}$$

(A) $i_r, j_r, k_r \in J, \alpha, \beta, \gamma \geqslant 0, \alpha + \beta + \gamma + l + m = a$ (A.9)

(B) if $j_r = d, j_s = \bar{d}$ for some $b < d \leqslant n$, then $n + r - s \geqslant d$ (A.10)

(C) one of the following (C1)–(C4) holds: (A.11)

(C1): $l = m \geqslant 1, l + \beta = n - b + 1$, i.e $\Bigl(\ldots, \overbrace{\underline{b \ldots b}b}^{l}, \ldots, \overbrace{\bar{b}\underline{\bar{b} \ldots \bar{b}}}^{l}, \ldots\Bigr)$

(C2): $l = m \geqslant 1, l + \beta = n - b + 2,$ i.e $\Big(\ldots, \overbrace{\underline{b \ldots b}}^{l}, \ldots, \overbrace{\underline{\bar{b} \ldots \bar{b}}}^{l}, \ldots\Big)$

(C3): $l = m + 1 \geqslant 1, l + \beta = n - b + 2,$ i.e $\Big(\ldots, \overbrace{\underline{b \ldots b}b}^{l}, \ldots, \overbrace{\underline{\bar{b} \ldots \bar{b}}}^{l-1}, \ldots\Big)$

(C4): $l = m - 1 \geqslant 0, l + \beta = n - b + 1,$ i.e $\Big(\ldots, \overbrace{\underline{b \ldots b}}^{l}, \ldots, \overbrace{\bar{b}\underline{\bar{b} \ldots \bar{b}}}^{l+1}, \ldots\Big)$

where the underlines indicate those b and $\bar{b}$ that are changed into $b-1$ and $\overline{b-1}$ respectively under the action of τ_b for $b \geqslant 2$. In short, condition (C) is selecting the tableaux of the form

$$(\ldots, \underline{b \ldots b}(b), \ldots, (\bar{b})\underline{\bar{b} \ldots \bar{b}}, \ldots) \tag{A.12}$$

where (b) and $(\bar{b})$ can be present or absent independently. Condition (B) says that (2.8) is satisfied for the segment $j_1 \prec \cdots \prec j_\beta$. In (A.8), the inequalities involving b or $\bar{b}$ should be imposed even when $l = 0$ or $m = 0$. We set

$$V_b = \bigsqcup_{l,m \geqslant 0, (l,m) \neq (0,0)} V_b^{l,m} \qquad 2 \leqslant b \leqslant n$$

where we assume $V_b^{l,m} = \emptyset$ unless $|l - m| = 0, \pm 1$.

As an example, we have

$$V = V_n = V_n^{1,1} \sqcup V_n^{1,2} \sqcup V_n^{2,1} \sqcup V_n^{2,2} \qquad V_n^{1,1} \subset W.$$

Given $3 \leqslant a \leqslant n$, only $V_n, V_{n-1}, \ldots, V_{n-a+2}$ are non-empty, and the last one reads

$$\begin{aligned}
&V_{a^\sharp} = V_{a^\sharp}^{1,1} \sqcup V_{a^\sharp}^{0,1} \sqcup V_{a^\sharp}^{1,0} && a^\sharp = n - a + 2 \\
&V_{a^\sharp}^{1,1} = \{(a^\sharp \prec j_1 \prec \cdots \prec j_{a-2} \prec \overline{a^\sharp})\} && \\
&V_{a^\sharp}^{0,1} = \{(j_1 \prec \cdots \prec j_{a-1} \prec \overline{a^\sharp})\} && (a^\sharp \prec j_1) \\
&V_{a^\sharp}^{1,0} = \{(a^\sharp \prec j_1 \prec \cdots \prec j_{a-1})\} && (j_{a-1} \prec \overline{a^\sharp})
\end{aligned}$$

where j_r obey condition (B). It is easy to see

$$\tau_{a^\sharp}(t) = t \qquad \text{for any } t \in V_{a^\sharp}. \tag{A.13}$$

Note that $a^\sharp \geqslant 2$.

Lemma A.3. *If $t \in V_b$, then $\tau_b(t) = t$ or $\tau_b(t) \in V_{b-1}$ for $3 \leqslant b \leqslant n$.*

Proof. We set $c = b - 1$. As in (A.12), an element $t \in V_b$ has the form $t = (\ldots, d, \underline{b \ldots b}(b), \ldots, (\bar{b})\underline{\bar{b} \ldots \bar{b}}, e, \ldots)$, where $d \prec b, \bar{b} \prec e$. Suppose $\tau_b(t) \neq t$. We classify the non-trivial action of τ_b into four cases; (D1) $d = c, e = \bar{c}$, (D2) $d = c, e \succ \bar{c}$, (D3) $d \prec c, e = \bar{c}$ and (D4) $d \prec c, e \succ \bar{c}$. In each case, the action $t \to \tau_b(t)$ is given as follows:

(D1): $(\ldots, c, \underline{b \ldots b}(b), \ldots, (\bar{b})\underline{\bar{b} \ldots \bar{b}}\bar{c}, \ldots) \mapsto (\ldots, \underline{cc \ldots c}(b), \ldots, (\bar{b})\underline{\bar{c} \ldots \bar{c}}\,\bar{c}, \ldots)$

(D2): $(\ldots, c, \underline{b \ldots b}(b), \ldots, (\bar{b})\underline{\bar{b} \ldots \bar{b}}, \ldots) \mapsto (\ldots, \underline{cc \ldots c}c(b), \ldots, (\bar{b})\underline{\bar{c} \ldots \bar{c}}, \ldots)$

(D3): $(\ldots, \underline{b \ldots b}(b), \ldots, (\bar{b})\underline{\bar{b} \ldots \bar{b}}\bar{c}, \ldots) \mapsto (\ldots, \underline{c \ldots c}(b), \ldots, (\bar{b})\bar{c}\,\underline{\bar{c} \ldots \bar{c}}\,\bar{c}, \ldots)$

(D4): $(\ldots, \underline{b \ldots b}(b), \ldots, (\bar{b})\underline{\bar{b} \ldots \bar{b}}, \ldots) \mapsto (\ldots, \underline{c \ldots c}c(b), \ldots, (\bar{b})\bar{c}\,\underline{\bar{c} \ldots \bar{c}}, \ldots)$.

Here the underlines on the RHS (respectively LHS) designate those $c, \bar{c}$ (respectively $b, \bar{b}$) changed under τ_c (respectively τ_b). Comparing (A.12) with the RHSs of (D1)–(D4), we see that $\tau_b(t)$ satisfies condition (C) (A.11) for V_c. Condition (A) is clear. Condition (B) is non-trivial only for the pair $(b, \bar{b})$ when (b) and $(\bar{b})$ are both present. In such cases the number β of the letters between (b) and $(\bar{b})$ in (D1)–(D4) is bounded by $\beta \leqslant n - b - 1$. Thus $n - \beta \geqslant b + 1 > b$, showing that (B) is valid. □

Given $t = t_n \in V = V_n$, let $t_d = \tau_{d+1} \cdots \tau_n(t)$. From (A.13) and lemma A.3, we deduce

Lemma A.4. *There exists a unique p such that*

$$a^\sharp \leqslant p \leqslant n \qquad t_p \neq t_{p+1} \neq \cdots \neq t_n \qquad t_d \in V_d \qquad \text{for} \quad p \leqslant d \leqslant n \qquad t_p = t_{p-1}.$$

Lemma A.5. $t_p \in W$.

Proof. In (D1)–(D4), the number of underlined letters on the RHS is fewer than that on the LHS only for (D4). Therefore the situation

$$t_{p+1} \overset{\tau_{p+1}}{\longmapsto} t_p \overset{\tau_p}{\longmapsto} t_{p-1} = t_p$$

means that the map τ_{p+1} underwent the pattern (D4) with length one underlines in its LHS. Namely we have

$$\left(\ldots, d, \underline{p+1}\, \overbrace{(p+1), \ldots, (\overline{p+1})}^{n-p}\, \underline{\overline{p+1}}, e, \ldots\right) = t_{p+1} \in V_{p+1}$$
$$\overset{\tau_{p+1}}{\longmapsto} \left(\ldots, d, p\, \overbrace{(p+1), \ldots, (\overline{p+1})}^{n-p}\, \bar{p}, e, \ldots\right) = t_p \in V_p \qquad \text{(A.14)}$$

where $d \preceq p-1$ and $\overline{p-1} \preceq e$. Let us check whether the RHS belongs to W. (Irrespectively of the presence or absence of $(p+1)$ and $(\overline{p+1})$, (A.14) says that there are always $n-p$ letters between p and $\bar{p}$.) First, there is no repetition of the same letter in the tableau t_p because of the definition of V_p. Second, the $(p, \bar{p})$ pair in the centre certainly breaks condition (2.8). □

Remark A.6. By definition (A.7), any tableau in W contains a pair $(q, \bar{q})$ breaking (2.8). Let us call such a pair with the largest value of $1 \leqslant q \leqslant n$ the *maximal breaking* pair. (Actually a pair $(1, \bar{1})$ can never break (2.8).) At the end of the proof of lemma A.5, we have also established the following: the maximal breaking pair of t_p is $(p, \bar{p})$.

Lemma A.7. *Suppose that in the tableau*

$$\left(\ldots, q, \overbrace{\cdots}^{\gamma}, \bar{q}, \ldots\right) \in W$$

the pair $(q, \bar{q})$ is the maximal breaking one. Then we have $\gamma = n - q$.

Proof. Since $(q, \bar{q})$ breaks (2.8), we know $q \geqslant n - \gamma$. Assume that $q > n - \gamma$, hence $\sharp\{q+1, q+2, \ldots, n\} = n - q < \gamma$. Then there is at least one pair $(r, \bar{r})$ with $q < r \leqslant n$ between q and $\bar{q}$, hence the tableau looks like

$$\left(\ldots, q, \overbrace{\cdots}^{\alpha}, r, \overbrace{\cdots}^{\delta}, \bar{r}, \overbrace{\cdots}^{\beta} \bar{q}, \ldots\right) \qquad \gamma = \alpha + \beta + \delta + 2.$$

By the definition, the pair $(r, \bar{r})$ must satisfy condition (2.8), meaning $r < n - \delta$. When there are more than one such r, we take the smallest one among these, which implies $\alpha + \beta \leqslant r - q - 1$. Now these relations lead to the contradiction:

$$0 < q - n + \gamma = q - n + \alpha + \beta + \delta + 2 \leqslant -n + \delta + r + 1 \leqslant 0. \qquad □$$

Note the consistency of (A.14), remark A.6 and lemma A.7.

By virtue of lemma A.4 and lemma A.5, we are entitled to define

$$\tau\colon\quad V \longrightarrow W \qquad t \longmapsto t_p = \tau_{p+1}\tau_{p+2}\cdots\tau_n(t)$$

where p is specified in lemma A.4. In example A.2, the LHS is an element of V. When calculating its image under τ, one has $p = 4$, and the answer is (A.4) and not (A.5). Observe that $(4, \bar{4})$ is certainly the maximal breaking pair in (A.4) containing $n - 4 = 5$ letters in between.

Proposition A.8. *The map $\tau\colon V \longrightarrow W$ is a bijection.*

Proof. We have only to construct the inverse of τ. For $3 \leqslant b \leqslant n$ we define the map σ_b : $(i_1, \ldots, i_a) \mapsto (i'_1, \ldots, i'_a)$ by making the transformation $\big(\ldots, b-1, \overbrace{\cdots}^{n-b+1}, \overline{b-1}, \ldots\big) \mapsto \big(\ldots, b, \overbrace{\cdots}^{n-b+1}, \bar{b}, \ldots\big)$ for all the $(b-1, \overline{b-1})$ pairs matching this configuration. Given any tableau $s \in W$, we set $\sigma(s) = \sigma_n \cdots \sigma_{p+2}\sigma_{p+1}(s)$, where p is determined from the condition that $(p, \bar{p})$ is the maximal breaking pair of s. By construction it is then evident that $\sigma(\tau(t)) = t$, $\tau(\sigma(s)) = s$ for any $t \in V$ and $s \in W$. □

Appendix B. Basic lemmas

We keep notation (2.20) but do not assume (2.19) and (2.21) in lemma B.1. Set

$$\tilde{x}_m(u) = \frac{[0, \ldots, m-1][2, \ldots, m]}{[1, \ldots, m][1, \ldots, m-1]} \qquad 1 \leqslant m \leqslant N. \tag{B.1}$$

We define $\tilde{e}_a(u)$ by (2.15) by replacing $x_m(u)$ with $\tilde{x}_m(u)$. In particular, $\tilde{e}_a(u) = 0$ if $a > N$ or $a < 0$.

Lemma B.1. ([**NNSY**]). *Given the integers $0 = i_0 < i_1 < \cdots < i_{N-1}$, let $\mu = (\mu_j)$ be the Young diagram with depth less than N specified by $\mu_j = i_{N-j} + j - N$. Let $\mu' = (\mu'_j)$ denote the transpose of μ. Assume $m \geqslant i_{N-1} - N + 1$. Then*

$$\begin{aligned}\frac{[0, i_1, i_2, \ldots, i_{N-1}]}{[m, \ldots, m+N-1]} &= \sum_t \prod_{(\alpha,\beta)\in(m^N)/\mu} \tilde{x}_{t(\alpha,\beta)}(u + \alpha + \beta - 2) \\ &= \det_{1\leqslant j,l\leqslant m}\left(\tilde{e}_{N-\mu'_j-l+j}\left(u + \frac{N-2+j+l-\mu'_j}{2}\right)\right)\end{aligned}$$

where the sum $\sum_t$ extends over the semistandard tableaux on the skew Young diagram $(m^N)/\mu$ [M1] on letters $\{1, \ldots, N\}$. $t(\alpha, \beta)$ denotes the entry of t at the αth row and the βth column from the bottom left corner.

The lemma is related to the ninth variation of the Schur function [M2], and applicable to q-characters for $U_q\big(A^{(1)}_{N-1}\big)$. It is actually valid for any $N \in \mathbb{Z}_{\geqslant 1}$. To approach the $U_q\big(C^{(1)}_n\big)$ case in question, we next take constraints (2.19) and (2.21) into account. We introduce the difference operators $L_j(u)(1 \leqslant j \leqslant N)$ by

$$L_j(u) = \overrightarrow{\prod_{i=N+1-j}^{N}} (D - \epsilon_i x_i(u + n + 1 - i)) \tag{B.2}$$

where $\epsilon_i = 1$ except $\epsilon_{n+1} = \epsilon_{n+2} = -1$. By lemma 2.2 and (2.9) we have $L(u) = L_N(u)$. Take the basis $\{w_1(u), \ldots, w_N(u)\}$ of the solutions to (2.19) such that

$$L_j(u)w_m(u) = 0 \qquad 1 \leqslant m \leqslant j \leqslant N. \tag{B.3}$$

Lemma B.2. *Under the above choice of the basis, we have $\tilde{x}_m(u) = x_m(u)$, where the latter is defined in (2.4) and (2.5).*

Proof. By calculating (B.2) directly, one gets ($1 \leqslant j \leqslant N-1$)

$$L_j(u) = D^j + (-1)^j \sigma'_j \frac{q_j(u+1)}{q_j(u)} + \text{terms proportional to } D, \ldots, D^{j-1} \tag{B.4}$$

$$\sigma'_j = \begin{cases} 1 & 1 \leqslant j \leqslant n+1 \\ -1 & n+2 \leqslant j \leqslant N-1 \end{cases}$$

$$q_j(u) = \begin{cases} Q_j\left(u + \frac{j-1}{2}\right) & 1 \leqslant j \leqslant n-1 \\ Q_n\left(u + \frac{n}{2}\right) Q_n\left(u + \frac{n-2}{2}\right) & j = n \\ Q_n\left(u + \frac{n}{2}\right)^2 & j = n+1 \\ Q_n\left(u + \frac{n}{2}\right) Q_n\left(u + \frac{n+2}{2}\right) & j = n+2 \\ Q_{N-j}\left(u + \frac{j-1}{2}\right) & n+3 \leqslant j \leqslant N-1. \end{cases}$$

Let $[0, \ldots, j-1]$ be the Casorati determinant of $w_m(u)$ as defined in (2.20). Due to (B.3) and (B.4) it satisfies the first order linear difference equation

$$\left(D - \sigma'_j \frac{q_j(u+1)}{q_j(u)}\right)[0, \ldots, j-1] = 0 \qquad 1 \leqslant j \leqslant N-1.$$

Thus we may set

$$[0, \ldots, j-1] = \phi_j(u) q_j(u) \tag{B.5}$$

where $\phi_j(u)$ is any function satisfying $\phi_j(u+1) = \sigma'_j \phi_j(u)$. Substituting (B.5) into (B.1) one finds $\tilde{x}_m(u) = x_m(u)$. □

Due to lemma B.2, we may set $\tilde{e}_a(u) = e_a(u)$. By combining proposition 2.4, (2.14) and (2.17), this can be further identified with $T_1^{(a)}(u)$ for all $a \in \mathbb{Z}$. Substituting this back to lemma B.1 and using (2.17), we obtain

Proposition B.3. *Let $i_0, \ldots, i_{N-1}, \mu$ and μ' be as in lemma B.1. Assume further that $w_1, \ldots, w_N$ satisfy (B.3). Then we have*

$$\frac{[0, i_1, i_2, \ldots, i_{N-1}]}{[0, \ldots, N-1]} = (-1)^{\mu_1} \sum_t \prod_{(\alpha,\beta) \in (\mu_1^N)/\mu} x_{t(\alpha,\beta)}(u + \alpha + \beta - 2)$$
$$= \det_{1 \leqslant j,l \leqslant \mu_1} \left(T_1^{(\mu'_j - j + l)} \left(u + \frac{N - 2 + j + l - \mu'_j}{2} \right) \right)$$

where the sum $\sum_t$ extends over the semistandard tableaux on the skew Young diagram $(\mu_1^N)/\mu$. $t(\alpha, \beta)$ is the entry of t at the αth row and the βth column from the bottom left corner of the skew Young diagram $(\mu_1^N)/\mu$.

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